Cheshire Cat Resurgence and Quasi-Exact Solvability

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References: arXiv:1609.06198[hep-th]

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Motivation: Asymptotic analysis & Nonperturbative physics

Perturbation theory in quantum mechanics

Eigenvalue problem of the Hamiltonian

$$\hat{H} = \frac{p^2}{2} + \frac{x^2}{2} + gV(x).$$

In most cases, we cannot solve $E(g)$.
$\Rightarrow$ Perturbation theory

$$E(g) \sim E_0 + E_1 g^1 + E_2 g^2 + \cdots .$$

Purpose of this talk

We would like to get a better understanding of perturbation series to compute nonperturbative phenomenon.

(cf. Berezin, Parisi, Zinn-Justin,, 1977, Stone, Reeve, 1978)

Asymptotic nature of perturbation theory

Typically, the perturbative coefficients of quantum mechanics show

$$E_n \sim \frac{n!}{A^n},$$

(because $\#$ of Feynman graphs $\sim n!$).

The convergence radius R of the perturbation theory is

$$R = \lim_{n\to\infty} \frac{|E_n|}{|E_{n+1}|} = \lim_{n\to\infty} \frac{A}{n+1} = 0.$$

This means that

- the perturbation series is divergent, but
- it works at least practically in many situations.

Error of the truncated perturbation series

To get a finite answer, we truncate the series at a certain order n:

$$\mathrm{Error}(n) \simeq E_n g^n \simeq \frac{n!}{A^n} g^n.$$

Optimal n would satisfy

$$\frac{\partial}{\partial n}\mathrm{Error}(n) = 0 \quad \Rightarrow \quad n_* = \frac{A}{g}.$$

Error at optimal n is thus exponentially small!

$$\mathrm{Error}(n_*) \simeq \exp\left(-\frac{A}{g}\right).$$

This formula is somewhat similar to nonperturbative corrections.

Trans-series and Resurgence

Beyond the perturbative expansion (trans-series expansion):

$$E(g) = \sum_{k=0}^{\infty} \mathrm{e}^{-kA/g} \sum_{n=0}^{\infty} E_{n,k} g^n.$$

Nonzero k's represent nonperturbative corrections.

Resurgence relation

$E_{n,0}$ at $n \gg 1$ already knows a part of nonperturbative physics,

$$E_{n,0} \sim \frac{n!}{A^n}.$$

Large order growth of one sector communicates with low order series of other sectors. (For more precise statements, see Delabaere, Pham, 1997, Berry, Howls, 1991, etc.)

Detailed plan of the talk

Summary of introduction

- $E(g) \sim E_0 + E_1 g + E_2 g^2 + \cdots$ is divergent.
- However, the optimal error is $O(\exp(-A/g))$.
- $E_n\ (n \gg 1)$ know about a part of missing nonpertubative corrections.

Purpose of this talk

Using the idea of resurgence theory,

- we solve a puzzle of semiclassical analysis about dynamical breaking/non-breaking of SUSY or Quasi-Exact Solvability (QES).
- we compute nonperturbative corrections to energies of the dynamically broken SUSY or QES.

Puzzles: Puzzles in semiclassical analysis of SUSY/QES systems

SUSY/QES quantum mechanics

We consider the Lagrangian,

$$\mathcal{L} = \frac{1}{g}\left(\frac{\dot{x}^2}{2} + \frac{(W'(x))^2}{2}\right) + \sum_{i=1}^{N_f} \overline{\psi}_i \left(\partial_t + W''(x)\right) \psi_i .$$

At $N_f = 1$, this Lagrangian is supersymmetric.
Integrating out fermions, one gets a bosonic effective theory,

$$\mathcal{L} = \frac{1}{g}\left(\frac{\dot{x}^2}{2} + \frac{(W'(x))^2}{2}\right) + \frac{N_f}{2} W''(x).$$

For integer N_f and special $W(x)$, this system is called Quasi-Exactly Solvable (QES).

Exact properties of SUSY sine-Gordon model

Let us set $W(x) = -\omega \cos(x)$ (and $N_f = 1$).
Because of unbroken SUSY, the ground state energy is

$$E(g) = 0.$$

In computing $E(g) \sim \sum_n E_n g^n$, the bosonic and fermionic contributions exactly cancel out:

$$E_n = 0.$$

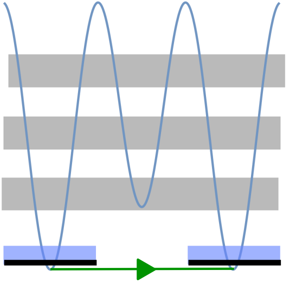
What happens when we go beyond the perturbation theory?

Instanton-type calculus

Let us perform the instanton calculus to go beyond the perturbation theory:

$$E(g) = \sum_n 0\, g^n - \exp(-2S_{\mathrm{inst}})(E_{0,1} + E_{1,1} g + E_{2,1} g^2 + \cdots) + \cdots .$$

Indeed, one can easily find the instanton (real-bion) configuration for SUSY sine-Gordon potential:



Puzzles in the instanton-type calculus

Exact computation

The SUSY sine-Gordon model has the ground state energy

$$E(g) = 0.$$

More generally, $E(g) =$ algebraic function in terms of g for $N_f = 1, 3, 5, \ldots$ thanks to QES.

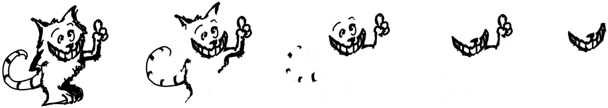
The nonperturbative correction is absent.

Instanton calculus

Semiclassical computation indicates the presence of the nonperturbative correction from real bions,

$$E(g) = -\mathrm{e}^{-2S_{\mathrm{inst}}/g}(E_{0,1} + O(g)) + \cdots (< 0).$$

Solution: Complex bions and Cheshire Cat Resurgence



(Cartoon by Roman Sulejmanpasic)

Path integral with complex classical solutions

We are now trying to evaluate (as $\beta \to \infty$)

$$\exp(-\beta E) = \int \mathcal{D}x(t) \exp\left(-\int_0^\beta \mathrm{d}t\, \mathcal{L}\right).$$

Using the perturbation method, E suffers from an error $O(\mathrm{e}^{-\#/g})$.

Origin of the error

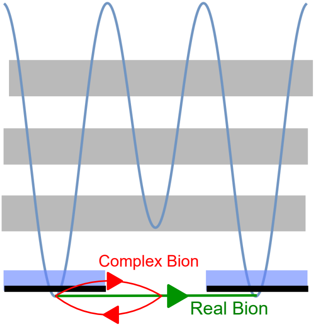
Instead of real spacetime paths $x(t) : \mathbb{R} \to \mathbb{R}$, there are a lot of complex classical solutions $x_\sigma(t) : \mathbb{R} \to \mathbb{C}$.

Contributions from x_σ to the semiclassical expansion are missed and create the error.

Real and complex bions

There are two kinds of classical solutions for Double sine-Gordon model:

$$\mathcal{L} = \frac{1}{g}\left(\frac{\dot{x}^2}{2} + \frac{(\omega \sin(x))^2}{2}\right) + \frac{N_f}{2}\omega \cos(x).$$



(Behtash, Sulejmanpasic, Schäfer , Ünsal, PRL 115 (2015) , 041601)

Classical actions of real and complex bions

The classical actions of Real and Complex bions (at $|g| \ll 1$) are

$$\begin{aligned} S_{\mathrm{RB}} &= 2S_{\mathrm{inst}}, \\ S_{\mathrm{CB}} &= 2S_{\mathrm{inst}} \pm \mathrm{i} N_f g \pi. \end{aligned}$$

Therefore, for $N_f = 1, 3, 5, \ldots$, the first nonperturbative correction to $E(g)$ becomes

$$\begin{aligned} E_{\mathrm{n.p.}} &\sim -\mathrm{e}^{-S_{\mathrm{RB}}/g} - \mathrm{e}^{-S_{\mathrm{CB}}/g} \\ &= -\mathrm{e}^{-2S_{\mathrm{inst}}/g} - \mathrm{e}^{-2S_{\mathrm{inst}}/g \pm \mathrm{i} N_f \pi} \\ &= 0, \end{aligned}$$

if we can show that both real and complex bions contribute.

Does complex bion contribute?

To prove the contribution from complex bions, we want to use Resurgence relation.

Resurgence relation Factorial growth of $E(g) \sim \sum_n E_n g^n$ knows about other complex classical solutions.

Because of SUSY at $N_f = 1$, however,

$$E_n = (\text{bosons}) - (\text{fermions}) = 0.$$

Nontrivial cancellation of Feynman diagrams hide the resurgence relation.

Deformation of SUSY sine-Gordon model

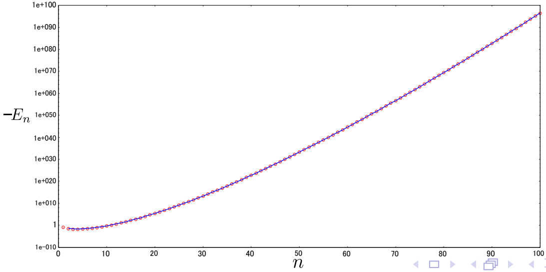
To break SUSY/QES, we deform $N_f \in \mathbb{Z}$ to $\zeta \in \mathbb{R}$:

$$\mathcal{L} = \frac{1}{g}\left(\frac{\dot{x}^2}{2} + \frac{(\sin(x))^2}{2}\right) + \frac{\zeta}{2}\cos(x).$$

If complex bion contributes, it predicts the following factorial growth

$$E_n \sim -\frac{1}{\pi}\frac{1}{8^{\zeta-1}}\frac{1}{\Gamma(1-\zeta)}\frac{(n-\zeta)!}{(2S_{\mathrm{inst}})^{n-\zeta+1}}.$$

Resurgence relation holds between trivial and complex-bion saddles:



Cheshire Cat Resurgence

Because of special properties at $\zeta = N_f = 1, 2, 3, \ldots$, the resurgence relation is hidden.
Only after a tiny deformation $\zeta \in \mathbb{R}$, we find

$$E_n \sim -\frac{1}{\pi}\frac{1}{8^{\zeta-1}}\frac{1}{\Gamma(1-\zeta)}\frac{(n-\zeta)!}{(2S_{\mathrm{inst}})^{n-\zeta+1}}.$$

$\Rightarrow$ Complex bions contribute to the semiclassical analysis at generic ζ.

Even after the limit $\zeta \to N_f$, this relation from resurgence must hold due to continuity.

$$\begin{aligned} E_{\mathrm{n.p.}} &\sim -\mathrm{e}^{-2S_{\mathrm{inst}}/g} - \mathrm{e}^{-2S_{\mathrm{inst}}/g \pm \mathrm{i}N_f\pi} \\ &= 0. \end{aligned}$$

Application: Nonperturbative corrections to pseudo-QES systems

Dynamically broken SUSY/pseudo-QES

We compute the ground-state energy of the tilted double-well potential:

$$\mathcal{L} = \frac{1}{g}\left(\frac{\dot{x}^2}{2} + \frac{(W'(x))^2}{2}\right) + \frac{N_f}{2} W''(x),$$

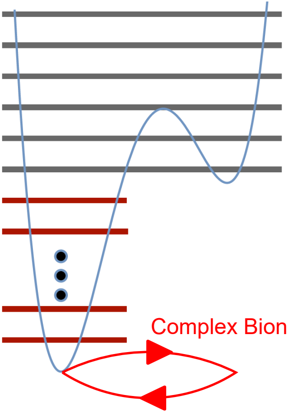
with

$$W(x) = \frac{x^3}{3} - \frac{\omega^2}{4} x.$$

At $N_f = 1$, the Lagrangian is supersymmetric, but the ground state breaks SUSY.
For general $N_f = 1, 2, 3, \ldots$, the perturbative part of $E(g)$ is solved algebraically but suffers from nonperturbative contribution (pseudo-QES).

Complex bions for tilted double Well potential

The double-well potential has a complex-bion solution but no real-bion solution:



Nonperturbative correction from complex bions

For simplicity, let us set $N_f = 1$. Due to SUSY, $E(g) \sim \sum_n 0 g^n$. Using Cheshire Cat resurgence, we can show that complex bions give the first nonperturbative correction.

$$E_{\mathrm{n.p.}} = -\frac{1}{2\pi}\left(\frac{g}{2}\right)^{N_f-1}\Gamma(N_f)\mathrm{e}^{-2S_{\mathrm{inst}}/g \pm \mathrm{i}N_f\pi}.$$

Setting $N_f = 1$, this correction is positive and it is consistent with the SUSY algebra $H = Q^\dagger Q \geq 0$.

Summary

- Asymptotic behaviors of the perturbation theory already indicates nonperturbative physics.
- For SUSY/(pseudo-)QES, the factorial growth of perturbation series is absent, but the tiny deformation revives it.
- Using the resurgence relation in the deformed theory, we show that complex bions contribute to the semiclassical analysis.
- We solved the puzzle about SUSY/QES in the semiclassical analysis, and evaluates $E(g)$ for broken SUSY/pseudo-QES.
- Stay tuned for the next week workshop!